

” Hamilton’s principle and the Generalized Field Theory : A comparative study ”

F. I. Mikhail

Department of mathematics, faculty of science, Ain shames university, **Egypt**

M. I. Wanas

Astronomy Department, Faculty of Science, Cairo university **Egypt**

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Abstract

The field equations of the generalized field theory (GFT) are derived from an action principle. A comparison between (GFT), Møller’s tetrad theory of gravitation (MTT), and general relativity is carried out regarding the Lagrangian of each theory. The results of solutions of the field equations, of each theory, are compared in case of spherical symmetry. The differences between the results are discussed and interpreted.

1 Introduction:

In the last fifteen years several field theories appeared in literature, each of which is assumed to be a natural generalization of the theory of general relativity. It is high time to carry out a thorough comparison between these theories with regard to three fundamental pivots of any generalized field theory, i.e:

- (i) the geometric structure used in the formalism of the theory,
- (ii) the procedure used in the derivation of its field equations,
- (iii) the models to which these field equations are applied and the results, of physical interest, obtained.

We believe that such comparison will be more feasible if we are able to unify these three aspects in the theories under consideration.

The present work is a trail to carry out such a comparison along the lines specified above. Strictly speaking, we are going to examine the two generalized field theories :

- (a) the generalized field theory (GFT) constructed by Mikhail & Wanas (1977),
- (b) Møller's tetrad theory of gravitation (MTT) (1978),

versus two different versions of general relativity (GR): the standard orthodox theory in which the lorentz signature is being imposed on the metric used from the beginning, and a modified version given by Wanas (1990) in which the Lorentz signature is introduced at a later stage.

Fortunately, the two theories (a), (b) under consideration depend on a Absolute Parallelism (Ap-Space) in their formalism. Thus the pivot(i) mentioned above is already unified. To satisfy pivot(ii) we have to derive the field equations of (GFT) using a variational principle, i.e. Hamilton's action principle. This is given in section 4 below. Besides, for comparison with (GR), we are going to use a Lagrangian function given by Møller (1978) as shown in section 6.

With regards to pivot (iii), we are going to apply the two theories (a),(b) and both versions of (GR) to the same absolute parallelism (AP) space, namely the most general AP-Space having spherical symmetry derived by Robertson (1932). The results obtained by applying the four different theories to this same AP-space are compared and tabulated in section 7.

2 Basic Geometric Structure (AP-space)

We are mainly interested in field theories using for their structure a geometry admitting absolute parallelism (AP-geometry). The AP-space is a 4-dimensional vector space (T_4), each point of which is labelled by 4-independent variables x^ν ($\nu = 0, 1, 2, 3$). At each point, 4-linearly independent contravariant vectors λ^μ_i , ($i = 0, 1, 2, 3$), are defined. Assuming that the $\lambda_{\mu i}$, is the normalized cofactor of λ^μ_i in the determinant $||\lambda^\mu_i||$. Hence they satisfy the relations,

$$\lambda^\mu_i \lambda_\mu_j = \delta_{ij} \quad (2.1)$$

$$\lambda^\mu_i \lambda_\nu_i = \delta^\mu_\nu \quad (2.2)$$

In what follows we use Greek letters ($\mu, \nu, \alpha, \beta, \dots$) to indicate coordinate components (world indices), and Latin letters (i, k, \dots) to indicate vector numbers (mesh indices). Summation convention will be applied to both types of indices. Using these vectors we can define the following 2nd order tensor:

$$\hat{g}_{\nu\mu} \stackrel{\text{def}}{=} \lambda_\nu_i \lambda_\mu_i, \quad (2.3)$$

$$\hat{g}^{\nu\mu} \stackrel{\text{def}}{=} \lambda^\nu_i \lambda^\mu_i. \quad (2.4)$$

It can be shown that

$$\hat{g}^{\mu\alpha} g_{\nu\alpha} \stackrel{\text{def}}{=} \delta^\mu_\nu, \quad (2.5)$$

$$\hat{g} = \hat{\lambda}^2, \quad (2.6)$$

where $\hat{g} \stackrel{\text{def}}{=} ||\hat{g}_{\mu\nu}||$, and $\hat{\lambda} \stackrel{\text{def}}{=} ||\lambda_\mu_i||$.

we can use $\hat{g}^{\mu\nu}$ and $\hat{g}_{\alpha\beta}$ to raise and lower world indices. It can also be used as a metric tensor defining a Riemannian space associated with the AP-space when needed. A non symmetric connection $\Gamma^\alpha_{\mu\nu}$ is then defined such that,

$$\lambda^\mu_{i+|\nu} \stackrel{\text{def}}{=} \lambda^\mu_{i,\nu} - \Gamma^\alpha_{\mu\nu} \lambda_\alpha_i = 0. \quad (2.7)$$

This is the condition of absolute parallelism. Equation (2.7) can directly be solved to give,

$$\Gamma^\alpha_{\mu\nu} = \lambda^\alpha_i \lambda_{\mu,\nu,i}. \quad (2.8)$$

It can be shown that (cf.Mikhail (1962)):

$$\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + \gamma^\alpha_{\mu\nu}, \quad (2.9)$$

where $\left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\}$ is Christoffel symbols defined as usual using the tensors (2.3), (2.4), and $\gamma^\alpha_{\mu\nu}$ is a 3rd order tensor defined by,

$$\gamma^\alpha_{\mu\nu} \stackrel{\text{def}}{=} \lambda^\alpha_i \lambda_{\mu;\nu,i}. \quad (2.10)$$

So,in the AP-space, we have three different connexions: $\left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\}$, $\Gamma^\alpha_{\mu\nu}$ and $\Gamma^\alpha_{(\mu\nu)} (= \frac{1}{2}(\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\nu\mu}))$. Consequently, we can define the following types of absolute derivatives:

$$A^\mu_{;\nu} \stackrel{\text{def}}{=} A^\mu_{,\nu} + \left\{ \begin{matrix} \mu \\ \nu\alpha \end{matrix} \right\} A^\alpha, \quad (2.11)$$

$$A^{\mu}_{+|\nu} \stackrel{\text{def}}{=} A^{\mu}_{,\nu} + \Gamma^{\mu}_{\alpha\nu} A^{\alpha}, \quad (2.12)$$

$$A^{\mu}_{-|\nu} \stackrel{\text{def}}{=} A^{\mu}_{,\nu} + \Gamma^{\mu}_{\nu\alpha} A^{\alpha}, \quad (2.13)$$

$$A^{\mu}_{|\nu} \stackrel{\text{def}}{=} A^{\mu}_{,\nu} + \Gamma^{\mu}_{(\alpha\nu)} A^{\alpha}, \quad (2.14)$$

where A^{μ} is an arbitrary contravariant vector and the comma $(,)$ denotes ordinary partial differentiation.

The torsion tensor is defined by

$$\Lambda^{\alpha}_{\mu\nu} \stackrel{\text{def}}{=} \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu} = -\Lambda^{\alpha}_{\nu\mu}. \quad (2.15)$$

Contracting this tensor by setting $\nu = \alpha$, we get the basic vector

$$C_{\mu} \stackrel{\text{def}}{=} \Lambda^{\alpha}_{\mu\alpha} = \gamma^{\alpha}_{\mu\alpha}. \quad (2.16)$$

For more details see Mikhail (1952).

3 Field Equations of (GFT):

Using the AP-space, as described in the above section, Mikhail & Wanas (1977) were able to construct a generalized field theory (GFT). Using a certain Lagrangian function, they were able to derive an identity of the form,

$$E^{\mu}_{+|\mu} = 0. \quad (3.1)$$

Following some analogy with (GR), they were led to chose their field equations into the form

$$E^{\mu}_{\nu} = 0, \quad (3.2)$$

where E^{μ}_{ν} is a non-symmetric tensor given by:

$$\begin{aligned} E^{\mu}_{\nu} \stackrel{\text{def}}{=} & \delta^{\mu}_{\nu} L - 2L^{\mu}_{\nu} - 2\delta^{\mu}_{\nu} C^{\alpha}_{|\alpha} \\ & - 2C^{\mu} C_{\nu} - 2\delta^{\mu}_{\alpha} C^{\beta} \Lambda^{\alpha}_{\beta\nu} + 2\hat{g}^{\alpha\mu} C_{+|\alpha} - 2\hat{g}^{\alpha\beta} \delta^{\mu}_{\epsilon} \Lambda^{\epsilon}_{\nu\beta +|\alpha}, \end{aligned} \quad (3.3)$$

and $L_{\mu\nu}$ is given by

$$\begin{aligned} L_{\mu\nu} \stackrel{\text{def}}{=} & \Lambda^{\alpha}_{\beta\mu} \Lambda^{\beta}_{\alpha\nu} - C_{\mu} C_{\nu}, \\ L \stackrel{\text{def}}{=} & \hat{g}^{\mu\nu} L_{\mu\nu}. \end{aligned} \quad (3.4)$$

Several promising solutions of this theory have been obtained, e.g. Wanas (1985, 1987, 1989).

However, it is of great interest to find out the particular lagrangian function which will give rise to the field equations (3.2) by using a variational principle. This is what we intend to do in the following section. In fact we were able to find out the Lagrangian function L giving rise to the scalar density,

$$\mathcal{L} \stackrel{\text{def}}{=} \hat{\lambda} L,$$

and then using the variational principle,

$$\delta \int \mathcal{L} d^4x = 0,$$

we will try to derive the field equations (3.2).

4 Derivation of the field equation of GFT using Hamilton's action principle

Generally speaking, starting with a Lagrangian function L which depends only on the tetrad vectors and their first coordinate derivatives i.e.

$$L = L(\lambda_\mu, \lambda_{\mu,\nu}), \quad (4.1)$$

we can define the scalar density,

$$\mathcal{L} \stackrel{\text{def}}{=} \hat{\lambda} L, \quad (4.2)$$

and thus :

$$\mathcal{L} = \mathcal{L}(\lambda_\mu, \lambda_{\mu,\nu}). \quad (4.3)$$

Then the action principle will give

$$\delta \int \mathcal{L}(\lambda_\mu, \lambda_{\mu,\nu}) d^4x = 0. \quad (4.4)$$

where $d^4x = dx^0 dx^1 dx^2 dx^3$. Then considering an arbitrary variation $(\delta \lambda_\mu)$ in the tetrad vectors, (4.4) can be put in the form,

$$\int \frac{\delta \mathcal{L}}{\delta \lambda_\mu} \delta \lambda_\mu d^4x = 0, \quad (4.5)$$

where

$$\frac{\delta \mathcal{L}}{\delta \lambda_\mu} \stackrel{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial \lambda_\mu} - \frac{\partial}{\partial x^\gamma} \left(\frac{\partial \mathcal{L}}{\partial \lambda_{\mu,\gamma}} \right), \quad (4.6)$$

is the Hamiltonian derivative of \mathcal{L} w.r.t. the arbitrary variation $(\delta \lambda_\mu)$. On the other hand we can write the Hamiltonian derivative (4.6) in the form,

$$\frac{\delta \mathcal{L}}{\delta \lambda_\mu} \stackrel{\text{def}}{=} \hat{\lambda} S_i^\mu. \quad (4.7)$$

It can be easily shown that S_i^μ is a vector for each value of i (Wanas 1975). Using S_i^μ , we can define the following 2nd order tensor,

$$S_\sigma^\beta \stackrel{\text{def}}{=} \lambda_\sigma S_i^\beta. \quad (4.8)$$

We assume that $(\delta \lambda_\beta)$ are linearly independent, and arbitrary as stated before, then for (4.5) to be satisfied, we write the Euler-Lagrange equation for the present problem :

$$\frac{\delta \mathcal{L}}{\delta \lambda_\mu} = \frac{\partial \mathcal{L}}{\partial \lambda_\beta} - \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}}{\partial \lambda_{\beta,\alpha}} \right) = 0. \quad (4.9)$$

Then using the same Lagrangian function defined by (3.4) in (GFT), the 1st term of (4.9) will take, after some manipulation, the form :

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \lambda_\beta} &= \hat{\lambda} \lambda_i^\beta \hat{g}^{\mu\nu} L_{\mu\nu} - \hat{\lambda} \left(\hat{g}^{\mu\beta} \lambda_i^\nu + \hat{g}^{\nu\beta} \lambda_i^\mu \right) L_{\mu\nu} \\ &\quad + \hat{\lambda} \hat{g}^{\mu\nu} \left(\lambda_i^\alpha \Lambda_{\mu\epsilon}^\beta \Lambda_{\alpha\nu}^\epsilon + \lambda_i^\epsilon \Lambda_{\epsilon\mu}^\alpha \Lambda_{\nu\alpha}^\beta - \lambda_i^\alpha \Lambda_{\alpha\mu}^\beta C_\nu - \lambda_i^\alpha \Lambda_{\alpha\nu}^\beta C_\mu \right). \end{aligned} \quad (4.10)$$

Similarly, we get for the 2nd term of (4.9) the expression :

$$\begin{aligned} \frac{\partial}{\partial x^\gamma} \frac{\partial \mathcal{L}}{\partial \lambda_{\beta,\gamma}} &= 2\hat{\lambda} \Gamma_{\mu\gamma}^\mu \left(\lambda_i^\epsilon \hat{g}^{\gamma\alpha} \Lambda_{\epsilon\alpha}^\beta - \lambda_i^\epsilon \hat{g}^{\alpha\beta} \Lambda_{\epsilon\alpha}^\gamma - \lambda_i^\gamma \hat{g}^{\alpha\beta} C_\alpha + \lambda_i^\beta \hat{g}^{\gamma\alpha} C_\alpha \right) \\ &\quad + 2\hat{\lambda} \left(\lambda_{i,\gamma}^\epsilon \hat{g}^{\gamma\alpha} \Lambda_{\epsilon\alpha}^\beta + \lambda_i^\epsilon \hat{g}^{\gamma\alpha} \Lambda_{,\gamma\epsilon\alpha}^\beta + \lambda_i^\epsilon \hat{g}^{\gamma\alpha} \Lambda_{\epsilon\alpha,\gamma}^\beta \right. \\ &\quad - \lambda_{i,\gamma}^\epsilon \hat{g}^{\alpha\beta} \Lambda_{\epsilon\alpha}^\gamma - \lambda_i^\epsilon \hat{g}^{\alpha\beta} \Lambda_{,\gamma\epsilon\alpha}^\gamma - \lambda_i^\epsilon \hat{g}^{\alpha\beta} \Lambda_{\epsilon\alpha,\gamma}^\gamma \\ &\quad \left. - \lambda_{i,\gamma}^\gamma C^\beta - \lambda_i^\gamma C_{,\gamma}^\beta - \lambda_{i,\gamma}^\beta C^\gamma + \lambda_i^\beta C_{,\gamma}^\gamma \right). \end{aligned} \quad (4.11)$$

Substituting from (4.10), (4.11) into Euler-Lagrange equation (4.9) we get, after comparing results with (4.7) & (4.8) the following set of field equations, (since $\hat{\lambda} \neq 0$ and λ_σ are linearly independent)

$$S_\sigma^\beta = 0, \quad (4.12)$$

where

$$\begin{aligned} S_\sigma^\beta &\stackrel{\text{def}}{=} \delta_\sigma^\beta L - 2L_\sigma^\beta + 2\hat{g}^{\mu\alpha} \Lambda_{\mu\nu}^\beta (\delta_\sigma^\epsilon \Lambda_{\epsilon\alpha}^\nu + \delta_\sigma^\nu C_\alpha) \\ &\quad - 2 \left[\Gamma_{\mu\gamma}^\mu \left(\delta_\sigma^\epsilon \hat{g}^{\gamma\alpha} \Lambda_{\epsilon\alpha}^\beta - \delta_\sigma^\epsilon \hat{g}^{\alpha\beta} \Lambda_{\epsilon\alpha}^\gamma - \delta_\sigma^\gamma \hat{g}^{\alpha\beta} C_\alpha + \delta_\sigma^\beta \hat{g}^{\gamma\alpha} C_\alpha \right) \right. \\ &\quad - \Gamma_{\sigma\gamma}^\epsilon \hat{g}^{\gamma\alpha} \Lambda_{\epsilon\alpha}^\beta + \delta_\sigma^\epsilon \hat{g}^{\gamma\alpha} \Lambda_{,\gamma\epsilon\alpha}^\beta + \delta_\sigma^\epsilon \hat{g}^{\gamma\alpha} \Lambda_{\epsilon\alpha,\gamma}^\beta - \Gamma_{\sigma\gamma}^\epsilon \hat{g}^{\alpha\beta} \Lambda_{\epsilon\alpha}^\gamma \\ &\quad \left. - \delta_\sigma^\epsilon \hat{g}^{\alpha\beta} \Lambda_{,\gamma\epsilon\alpha}^\gamma - \delta_\sigma^\epsilon \hat{g}^{\alpha\beta} \Lambda_{\epsilon\alpha,\gamma}^\gamma + \Gamma_{\sigma\gamma}^\gamma C^\beta - \delta_\sigma^\gamma C_{,\gamma}^\beta - \Gamma_{\sigma\gamma}^\beta C^\alpha + \delta_\sigma^\beta C_{,\gamma}^\gamma \right]. \end{aligned} \quad (4.13)$$

Using the following results (Wanas ((1975))

$$C_{|\sigma}^{\nu} - C_{|\sigma}^{+} = -C^{\epsilon} \Lambda_{\epsilon\sigma}^{\nu},$$

$$\hat{g}_{|\gamma}^{+\alpha} = 0 \quad , \quad \hat{g}_{|\sigma}^{\alpha\nu} = 0,$$

and the identity (Einstein (1929))

$$\Lambda_{++|\gamma}^{\gamma} = C_{+|\alpha}^{\sigma} - C_{+|\sigma}^{\alpha},$$

then (4.13) can be written in the form,

then (4.13) can be written in the form

$$\begin{aligned} S_{\sigma}^{\beta} &\stackrel{\text{def}}{=} \delta_{\sigma}^{\beta} L - 2L_{\sigma}^{\beta} - 2\delta_{\sigma}^{\beta} C_{|\gamma}^{\gamma} \\ &\quad - 2C^{\beta} C_{\sigma} - 2\delta_{\nu}^{\beta} C^{\epsilon} \Lambda_{\epsilon\sigma}^{\nu} + 2\hat{g}^{\alpha\beta} C_{+|\alpha}^{\sigma} - 2\hat{g}^{\gamma\alpha} \delta_{\nu}^{\beta} \Lambda_{++|\gamma}^{\nu}, \end{aligned} \quad (4.14)$$

which is identical with the tensor E_{ν}^{μ} given by (3.3) in the Mikhail- Wanas derivation. This shows that the field equations of (GFT) can be derived from an action principle with the Lagrangian function given by (3.4), (4.2).

5 Role of the signature

In metric theories of gravity, the Lorentz signature plays a very important role. The signature is defined as the difference between the number of +ve and -ve eigenvalues of the metric tensor. Lorentz signature is imposed on the metric, for purely physical reasons, namely to account for the special relativity in the limiting case. This reflects the fact that we distinguish between spatial and temporal sections of space-time. It was realized, since the appearance of special relativity, that our universe is a 4-dimensional one. So, there is no fundamental need to distinguish between space and time from the beginning. We are unable neither, to construct equipment or experiment, nor to measure in 4-dimensions, but rather in (3+1) dimensions. Thus, in order to compare the results of field theories with observation and experiment, Lorentz signature should be introduced at a later stage in the theory to reflect our distinction between space & time. In fact, the introduction of Lorentz signature transfers the theory from 4-dimensions to (3+1) dimensions.

Wanas (1990) suggested that the insertion of Lorentz signature is to be done just before matching the results of the theory with observations or experiments, i.e. after solving the field equations. He speculated that this may give rise to new physics, and give an example confirming his speculation.

There are different ways for inserting the Lorentz signature in a theory constructed in the AP-space. One way is just to change the sign of some constants of integration after solving the field equations. Another way is to replace (2.3) by

$$g_{\mu\nu} \stackrel{\text{def}}{=} e_i \lambda_{\mu} \lambda_{\nu}$$

where $e_i = (+1, -1, -1, -1)$ is the Levi-Civita indicator. A 3rd way is to take the vector λ^μ to be imaginary. For AP-space whose associated metric is diagonal (or it could be diagonalized), it doesn't matter whether we insert the Lorentz signature before or after solving the field equations.

6 Direct comparison between the four theories :

So far we have satisfied the two factors (i) & (ii) mentioned in the introduction. It may be of interest to compare the four theories, under consideration, with regard to the Lagrangian function used and the field equations derived in each.

For standard (GR) (written in the AP-space) the Lagrangian is usually taken to be (cf. Møller 1978),

$$\mathcal{L}_{GR} = \sqrt{-g} (\gamma^{\alpha\mu\nu} \gamma_{\nu\mu\alpha} - C^\nu C_\nu). \quad (6.1)$$

Consequently, the modified version of (GR) as speculated by Wanas (1990) will be,

$$\mathcal{L}_{\hat{GR}} = \sqrt{\hat{g}} (\gamma^{\alpha\mu\nu} \gamma_{\nu\mu\alpha} - C^\nu C_\nu). \quad (6.2)$$

The corresponding Lagrangian of (GFT), as given above,

$$\mathcal{L}_{GFT} = \sqrt{\hat{g}} (3\gamma^{\alpha\mu\nu} \gamma_{\nu\mu\alpha} - \gamma^{\alpha\mu\nu} \gamma_{\alpha\mu\nu} - C^\nu C_\nu). \quad (6.3)$$

Møller (1978) has chosen for his theory the Lagrangian function,

$$\mathcal{L}_{MTT} = \sqrt{-g} ((1 - 2\psi) \gamma^{\alpha\mu\nu} \gamma_{\nu\mu\alpha} + \psi \gamma^{\alpha\mu\nu} \gamma_{\alpha\mu\nu} - C^\nu C_\nu), \quad (6.4)$$

where ψ is a free parameter of order unity. It is clear that $\mathcal{L}_{GR}, \mathcal{L}_{GFT}$ can be obtained as special cases of Møller's Lagrangian (5.3) by taking,

$$\psi = 0 \quad , \quad \psi = -1,$$

respectively. However, although the lagrangian used by Møller in (MTT) appears to be more general than those for deriving the field equations of (GR), and (GFT), yet Møller's theory (MTT) as a whole is no so general. This will be shown clearly in the next section.

The reason, in our opinion, is due to the following two factors :

- (i) In Møller's theory, the Lorentz signature is imposed on the theory from the beginning.
- (ii) Møller used a phenomenological definition for the material-energy tensor $T_{\mu\nu}$, while it is defined geometrically in (GFT).

Formally, (6.1) & (6.2) are similar, but the results of application are different. We are going to consider the theory depending on (6.2) as different compared to (GR). The following table summarizes the main features of each theory.

Table 1: Comparison between the four field theories

Field Theory	Field Lagrangian	Field Equations	Field Variables	Gravitational Potential	$T_{\mu\nu}$
GR (1916)	\mathcal{L}_{GR}	$G_{\mu\nu} = -\kappa T_{\mu\nu}$	$g_{\mu\nu}$	$g_{\mu\nu}$	Phenom.
GFT (1977)	\mathcal{L}_{GFT}	$\hat{G}_{\mu\nu} = B_{\mu\nu}$ $E_{[\mu\nu]} = 0$	λ_i^μ	$g_{\mu\nu}$	Geomet.
MTT (1978)	\mathcal{L}_{MTT}	$G_{\mu\nu} + H_{\mu\nu} = -\kappa T_{\mu\nu}$ $V_{[\mu\nu]} = 0$	λ_i^μ	$g_{\mu\nu}$	Phenom.
$\hat{G}R$ (1990)	$\hat{\mathcal{L}}_{GR}$	$\hat{G}_{\mu\nu} = -\kappa T_{\mu\nu}$	$\hat{g}_{\mu\nu}$	$g_{\mu\nu}$	Phenom.

where $G_{\mu\nu}$ is Einstein tensor.

7 Comparison in the case of spherical symmetry

Since all theories tabulated in table 1 are written in the AP-space, it is convenient to compare them by using the most general AP-space having spherical symmetry. The structure of this space is given by Robertson (1932). The tetrad giving the structure of this space can be written in spherical polar coordinates as,

$$\lambda_i^\mu = \begin{pmatrix} A & Dr & 0 & 0 \\ 0 & B \sin \theta \cos \phi & \frac{B}{r} \cos \theta \cos \phi & -\frac{B \sin \phi}{r \sin \theta} \\ 0 & B \sin \theta \sin \phi & \frac{B}{r} \cos \theta \sin \phi & \frac{B \cos \phi}{r \sin \theta} \\ 0 & B \cos \theta & -\frac{B}{r} \sin \theta & 0 \end{pmatrix}, \quad (7.5)$$

where A,B,D are unknown functions of r only. Calculating the necessary tensors for this space we find that (see table 1):

(1) For Møller's tetrad theory $H_{\mu\nu} = 0$, $V_{[\mu\nu]} = 0$ is satisfied identically.

(2) For GFT $B_{\mu\nu} = 0$, $E_{[\mu\nu]} = 0$ is satisfied identically.

Furthermore, if we direct our attention to free space solutions ($T_{\mu\nu} = 0$) we found the results summarized in table 2.

Table 2: Comparison in the case of spherically symmetric solutions

Field Theory	Field equations	Schwarz. solution?	Other Solutions ?	Reference
GR	$G_{\mu\nu} = 0$	Yes	No	Birkhoff Theorem
GFT	$\hat{G}_{\mu\nu} = 0$	Yes	Yes	Wanas (1985)
MTT	$G_{\mu\nu} = 0$	Yes	No	Mikhail et. al. (1991)
$\hat{G}R$	$\hat{G}_{\mu\nu} = 0$	Yes	Yes	Wanas (1990)

It is clear from the second column in table(2) above that although the field equations of the four theories reduce formally to those of GR in free space, the results obtained are not identical. It is clear that this difference is a direct consequence to the stage at which Lorentz signature is introduced in the theory.

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